## Homological algebra solutions Week 3

1. Let us consider a sequence of chain complexes :

$$0 \xrightarrow{i_{\bullet}} A_{\bullet} \xrightarrow{\alpha_{\bullet}} B_{\bullet} \xrightarrow{\beta_{\bullet}} C_{\bullet} \xrightarrow{j_{\bullet}} 0$$

To show that the sequence is exact at  $B_{\bullet}$  in the abelian category  $Ch(\mathcal{A})$ , we need to show that

$$\beta_{\bullet} \circ \alpha_{\bullet} = 0_{\bullet}$$

and that the induced map

$$\operatorname{im}(\alpha_{\bullet}) \to \ker(\beta_{\bullet})$$

is an isomorphism of chain complexes. But since

$$0 \xrightarrow{i_n} A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{j_n} 0$$

is exact in  $\mathcal{A} \ \forall n \in \mathbb{Z}$ , we have that

$$(\alpha_{\bullet} \circ \beta_{\bullet})_n = \alpha_n \circ \beta_n = 0 \ \forall n \in \mathbb{Z}$$

Thus  $\alpha_{\bullet} \circ \beta_{\bullet} = 0_{\bullet}$ . Now again by exactness we get that

$$\operatorname{im}(\alpha_{\bullet})_n = \operatorname{im}(\alpha_n) \cong \ker(\beta_n) = \ker(\beta_{\bullet})_n \ \forall n \in \mathbb{Z}$$

hence the induced map

$$\operatorname{im}(\alpha_{\bullet}) \to \ker(\beta_{\bullet})$$

is an isomorphism.

We check similarly exactness at  $A_{\bullet}$  and  $C_{\bullet}$  and thus we conclude that the sequence of chain complexes is exact.

2. Consider the following diagram:

$$\dots \longrightarrow D_n \xrightarrow{h_n} B_n \xrightarrow{f_n} C_n \longrightarrow \dots$$

$$\downarrow^d \qquad \downarrow^d \qquad \downarrow^d \qquad \downarrow^d \qquad \downarrow^d \qquad \dots$$

$$\dots \longrightarrow D_{n-1} \xrightarrow{h_{n-1}} B_{n-1} \xrightarrow{f_{n-1}} C_{n-1} \longrightarrow \dots$$

where  $f_n \circ h_n = g_n \ \forall n \in \mathbb{Z}$ . Using that  $f_{\bullet}$  and  $g_{\bullet}$  are chain complexes morphisms we have that

$$f_{n-1} \circ d \circ h_n = d \circ f_n \circ h_n = d \circ g_n = g_{n-1} \circ d = f_{n-1} \circ h_{n-1} \circ d$$

but since  $f_{\bullet}$  is monic we conclude that

$$d \circ h_n = h_{n-1} \circ d$$

Thus  $h_{\bullet}$  is a chain complexes morphism such that  $f_{\bullet} \circ h_{\bullet} = g_{\bullet}$ .

3. Without loss of generality we prove for exact rows. First we recall that since  $D_{\bullet,\bullet}$  is a bounded double complex, we have

$$(\operatorname{Tot}(D_{\bullet,\bullet}))_n = \bigoplus_{p+q=n} D_{p,q}$$

and maps  $d = d^h + d^v : (\text{Tot}(D_{\bullet,\bullet})_n \to (\text{Tot}(D_{\bullet,\bullet}))_{n-1})$ . First we remark that

$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = (d^h \circ d^h) + (d^v \circ d^v) + (d^v \circ d^h + d^h \circ d^v) = 0$$

by definition of a double complex.

Now by Freyd-Mitchell Embedding Theorem we only need to prove that

$$im(d_n) = \ker(d_{n+1})$$

in the category of R-modules  $\forall n \in \mathbb{Z}$ . We know that

$$\operatorname{im}(d_n^h) = \ker(d_{n+1}^h) \ \forall n \in \mathbb{Z}$$

by exactness of the rows. Also by the previous computation

$$\operatorname{im}(d_n) \subseteq \ker(d_{n+1})$$

Thus we only need to show that

$$\ker(d_{n+1}) \subseteq \operatorname{im}(d_n)$$

But notice that

$$\alpha \in \ker(d_{n+1}) \implies \alpha \in \ker(d_{n+1}^v)$$

but since  $\ker(d_{n+1}^h) = \operatorname{im}(d_n^h)$  there exists  $\alpha_1$  such that  $d_n^h(\alpha_1) = \alpha$ . Now using the anti-commutativity  $d^h \circ d^v = -d^v \circ d^h$ , we get

$$d_{n+1}^{v}(\alpha) = 0 \iff d_{n+1}^{v} \circ d_{n}^{h}(\alpha_{1}) = 0 \iff -d_{n}^{h} \circ d_{n+1}^{v}(\alpha_{1}) = 0$$

which implies that

$$-d_{n+1}^v(\alpha_1) \in \ker(d_n^h) = \operatorname{im}(d_{n+1}^h)$$

i.e. again there exists  $\alpha_2$  such that

$$d_{n+1}^{h}(\alpha_2) = -d_{n+1}^{v}(\alpha_2)$$

We iterate the argument to construct  $\{\alpha_l\}_{l\geq 1}$  such that

$$d_{n+l-1}^h(\alpha_l) = -d_{n+l-1}^v(\alpha_l) \ \forall n \ge 1$$

But since  $D_{\bullet,\bullet}$  is bounded there must exists  $m \geq 1$  such that

$$d_{n+m-1}^v(\alpha_m) = 0 \implies \alpha_m = 0$$

Now observe that

$$d_n(\sum_{i=1}^m \alpha_i) = \sum_{i=1}^m d^v(\alpha_i) + \sum_{i=1}^m d^v(\alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha + \sum_{i=2}^m d^h(\alpha_i)$$

Which means by definition of  $\{\alpha_i\}_{1 \leq i \leq m}$ 

$$d(\sum_{i=1}^{m} \alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha - \sum_{i=1}^{m-1} d^v(\alpha_i) = \alpha + \alpha_m = \alpha$$

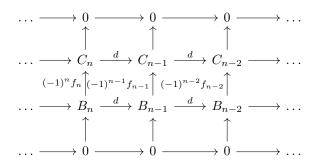
which shows that  $\ker(d) \subset \operatorname{im}(d)$  and thus concludes.

4. (a) We define the double complex  $D_{\bullet,\bullet}$  with only two non trivial rows  $B_{\bullet}$  and  $C_{\bullet}$  by :

where anti-commutativity of the non-trivial squares is given by f being a chain map.

Note that equivalently we could define  $D_{\bullet,\bullet}$  using more explicitly the

sign trick:



(b) Let  $A_{\bullet,\bullet}$  be a double complex with maps  $d^h_{p,q}:A_{p,q}\to A_{p-1,q}$  and  $d^v_{p,q}:A_{p,q}\to A_{p,q-1}$ . Using the sign trick we get a chain complex  $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$  with boundary maps  $\varphi_{\bullet,q}:A_{\bullet,q}\to A_{\bullet,q-1}$ , defined by

$$\varphi_{p,q} = (-1)^p d_{p,q}^v$$

Now applying again a sign trick to this chain complex  $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$  we define a double complex  $A'_{\bullet,\bullet}$  with maps

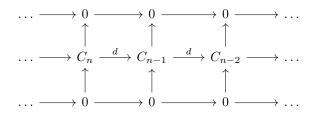
$$d_{p,q}^{\prime v} = (-1)^p \varphi_{p,q} \quad d_{p,q}^{\prime h} = d_{p,q}^h$$

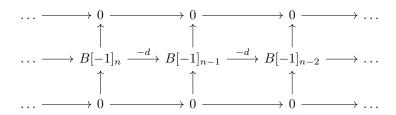
But

$$d_{p,q}'^v = (-1)^p \varphi_{p,q} = (-1)^p (-1)^p d_{p,q}^v = d_{p,q}^v$$

i.e.  $A'_{\bullet,\bullet} = A_{\bullet,\bullet}$  and thus the sign trick is a one to one correspondence between objects of  $\operatorname{Ch}(\operatorname{Ch}(\mathcal{A}))$  and double complexes.

(c) We can consider  $C_{\bullet}$  (resp.  $B[-1]_{\bullet}$ ) as a double complex with only one non trivial row in the following way





Now for the exactness of the short sequence, we denote

$$0 \longrightarrow C_{\bullet} \xrightarrow{\iota_{\bullet,\bullet}} D_{\bullet,\bullet} \xrightarrow{\pi_{\bullet,\bullet}} B[-1]_{\bullet} \longrightarrow 0$$

Setting the indices such that

$$C_{1,n} = C_n$$
  $D_{1,n} = C_n$   $D_{0,n} = B[-1]_{n+1}$   $B[-1]_{0,n} = B[-1]_n$ 

And  $\iota_{\bullet,\bullet}, \pi_{\bullet,\bullet}$  are defined as

$$\iota_{p,q} = \begin{cases} \operatorname{Id}_{C_q} & \text{if } p = 1\\ \operatorname{Id}_0 & \text{if } p \neq 1 \end{cases}$$

$$\pi_{p,q} = \begin{cases} \operatorname{Id}_{B_q} & \text{if } p = 0\\ \operatorname{Id}_0 & \text{if } p \neq 0 \end{cases}$$

From point b) this corresponds to a short sequence in  $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$  using the sign trick. From exercise 1, we only to check exactness at each non trivial rows i.e.

i.

$$\iota_{1,n} = \mathrm{Id}_{C_n} : C_{1,n} = C_n \to D_{1,n} = C_n$$

is an injective morphism of chain complexes.

ii.

$$\pi_{0,n} = \mathrm{Id}_{B_n} : D_{0,n} = B_n \to B[-1]_{n+1} = B_n$$

is a surjective morphism of chain complexes.

iii.

$$\iota_{1,n} \circ \pi_{1,n} = 0$$
  $\iota_{0,n} \circ \pi_{0,n} = 0$ 

i. and ii. are clear from the definitions of  $\iota_{1,n}$  and  $\pi_{0,n}$ . Finally iii. holds since

$$\iota_{0,n} = 0 \ \forall n \in \mathbb{Z} \ \text{and} \ \pi_{1,n} = 0 \ \forall n \in \mathbb{Z}$$

Note that the mapping cone of  $f: D_{\bullet} := \text{Tot}(D_{\bullet, \bullet})$  is given by :

$$D_{\bullet} = B[-1]_{\bullet} \oplus C_{\bullet}$$

with chain map

$$d_{D,\bullet} = \begin{pmatrix} -d_{B[-1],\bullet} & 0\\ f[-1]_{\bullet} & d_{C,\bullet} \end{pmatrix}$$

We check that it is indeed a chain complex :

$$d_{D,n} \circ d_{D,n-1} = \begin{pmatrix} -d_{B[-1],n} & 0 \\ f[-1]_n & d_{C,n} \end{pmatrix} \begin{pmatrix} -d_{B[-1],n-1} & 0 \\ f[-1]_{n-1} & d_{C,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} d_{B[-1],n} \circ d_{B[-1],n-1} & 0 \\ -f[-1]_n \circ d_{B[-1],n-1} + d_{C,n} \circ f[-1]_{n-1} & d_{C,n} \circ d_{C,n-1} \end{pmatrix} = 0$$

since  $f_{\bullet}: B_{\bullet} \to C_{\bullet}$  is a chain map.