Homological algebra solutions Week 3

1. Let us consider a sequence of chain complexes :

$$0 \xrightarrow{i_\bullet} A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} C_\bullet \xrightarrow{j_\bullet} 0$$

   To show that the sequence is exact at $B_\bullet$ in the abelian category $\mathrm{Ch}(\mathcal{A})$, we need to show that

$$\beta_\bullet \circ \alpha_\bullet = 0_\bullet$$

   and that the induced map

$$\mathrm{im}(\alpha_\bullet) \to \ker(\beta_\bullet)$$

   is an isomorphism of chain complexes. But since

$$0 \xrightarrow{i_n} A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{j_n} 0$$

   is exact in $\mathcal{A}$ $\forall n \in \mathbb{Z}$, we have that

$$(\alpha_\bullet \circ \beta_\bullet)_n = \alpha_n \circ \beta_n = 0 \; \forall n \in \mathbb{Z}$$

   Thus $\alpha_\bullet \circ \beta_\bullet = 0_\bullet$. Now again by exactness we get that

$$\mathrm{im}(\alpha_\bullet)_n = \mathrm{im}(\alpha_n) \cong \ker(\beta_n) = \ker(\beta_\bullet)_n \; \forall n \in \mathbb{Z}$$

   hence the induced map

$$\mathrm{im}(\alpha_\bullet) \to \ker(\beta_\bullet)$$

   is an isomorphism.
   We check similarly exactness at $A_\bullet$ and $C_\bullet$ and thus we conclude that the sequence of chain complexes is exact.

2. Consider the following diagram :

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & D_n & \xrightarrow{h_n} & B_n & \xrightarrow{f_n} & C_n & \longrightarrow & \dots \\
& & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
\dots & \longrightarrow & D_{n-1} & \xrightarrow{h_{n-1}} & B_{n-1} & \xrightarrow{f_{n-1}} & C_{n-1} & \longrightarrow & \dots
\end{array}$$

where $f_n \circ h_n = g_n\ \forall n \in \mathbb{Z}$. Using that $f_\bullet$ and $g_\bullet$ are chain complexes morphisms we have that

$$f_{n-1} \circ d \circ h_n = d \circ f_n \circ h_n = d \circ g_n = g_{n-1} \circ d = f_{n-1} \circ h_{n-1} \circ d$$

but since $f_\bullet$ is monic we conclude that

$$d \circ h_n = h_{n-1} \circ d$$

Thus $h_\bullet$ is a chain complexes morphism such that $f_\bullet \circ h_\bullet = g_\bullet$.

3. Without loss of generality we prove for exact rows. First we recall that since $D_{\bullet,\bullet}$ is a bounded double complex, we have

$$(\mathrm{Tot}(D_{\bullet,\bullet}))_n = \bigoplus_{p+q=n} D_{p,q}$$

and maps $d = d^h + d^v : (\mathrm{Tot}(D_{\bullet,\bullet})_n \to (\mathrm{Tot}(D_{\bullet,\bullet}))_{n-1}$. First we remark that

$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = (d^h \circ d^h) + (d^v \circ d^v) + (d^v \circ d^h + d^h \circ d^v) = 0$$

by definition of a double complex.
Now by Freyd-Mitchell Embedding Theorem we only need to prove that

$$\mathrm{im}(d_n) = \ker(d_{n+1})$$

in the category of R-modules $\forall n \in \mathbb{Z}$. We know that

$$\mathrm{im}(d_n^h) = \ker(d_{n+1}^h)\ \forall n \in \mathbb{Z}$$

by exactness of the rows. Also by the previous computation

$$\mathrm{im}(d_n) \subseteq \ker(d_{n+1})$$

Thus we only need to show that

$$\ker(d_{n+1}) \subseteq \mathrm{im}(d_n)$$

But notice that

$$\alpha \in \ker(d_{n+1}) \implies \alpha \in \ker(d_{n+1}^v)$$

but since $\ker(d_{n+1}^h) = \mathrm{im}(d_n^h)$ there exists α_1 such that $d_n^h(\alpha_1) = \alpha$. Now using the anti-commutativity $d^h \circ d^v = -d^v \circ d^h$, we get

$$d_{n+1}^v(\alpha) = 0 \iff d_{n+1}^v \circ d_n^h(\alpha_1) = 0 \iff -d_n^h \circ d_{n+1}^v(\alpha_1) = 0$$

which implies that

$$-d_{n+1}^v(\alpha_1) \in \ker(d_n^h) = \mathrm{im}(d_{n+1}^h)$$

i.e. again there exists α_2 such that

$$d^h_{n+1}(\alpha_2) = -d^v_{n+1}(\alpha_2)$$

We iterate the argument to construct $\{\alpha_l\}_{l\geq 1}$ such that

$$d^h_{n+l-1}(\alpha_l) = -d^v_{n+l-1}(\alpha_l)\ \forall n \geq 1$$

But since $D_{\bullet,\bullet}$ is bounded there must exists $m \geq 1$ such that

$$d^v_{n+m-1}(\alpha_m) = 0 \implies \alpha_m = 0$$

Now observe that

$$d_n(\sum_{i=1}^{m}\alpha_i) = \sum_{i=1}^{m} d^v(\alpha_i) + \sum_{i=1}^{m} d^v(\alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha + \sum_{i=2}^{m} d^h(\alpha_i)$$

Which means by definition of $\{\alpha_i\}_{1\leq i\leq m}$

$$d(\sum_{i=1}^{m}\alpha_i) = \alpha_m + \sum_{i=1}^{m-1} d^v(\alpha_i) + \alpha - \sum_{i=1}^{m-1} d^v(\alpha_i) = \alpha + \alpha_m = \alpha$$

which shows that $\ker(d) \subset \operatorname{im}(d)$ and thus concludes.

4. (a) We define the double complex $D_{\bullet,\bullet}$ with only two non trivial rows $B_\bullet$ and $C_\bullet$ by :

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & C_n \arrow[u] \arrow[r, "d"] & C_{n-1} \arrow[u] \arrow[r, "d"] & C_{n-2} \arrow[u] \arrow[r] & \dots \\
\dots \arrow[r] & B_n \arrow[u, "f_n"] \arrow[r, "-d"] & B_{n-1} \arrow[u, "f_{n-1}"] \arrow[r, "-d"] & B_{n-2} \arrow[u, "f_{n-2}"] \arrow[r] & \dots \\
\dots \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

where anti-commutativity of the non-trivial squares is given by f being a chain map.
Note that equivalently we could define $D_{\bullet,\bullet}$ using more explicitly the

sign trick :

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & C_n \arrow[r, "d"] \arrow[u] & C_{n-1} \arrow[r, "d"] \arrow[u] & C_{n-2} \arrow[r] \arrow[u] & \dots \\
\dots \arrow[r] & B_n \arrow[r, "d"] \arrow[u, "(-1)^n f_n"] & B_{n-1} \arrow[r, "d"] \arrow[u, "(-1)^{n-1} f_{n-1}"] & B_{n-2} \arrow[r] \arrow[u, "(-1)^{n-2} f_{n-2}"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

(b) Let $A_{\bullet,\bullet}$ be a double complex with maps $d^h_{p,q} : A_{p,q} \to A_{p-1,q}$ and $d^v_{p,q} : A_{p,q} \to A_{p,q-1}$. Using the sign trick we get a chain complex $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$ with boundary maps $\varphi_{\bullet,q} : A_{\bullet,q} \to A_{\bullet,q-1}$, defined by

$$\varphi_{p,q} = (-1)^p d^v_{p,q}$$

Now applying again a sign trick to this chain complex $\{A_{\bullet,q}\}_{q\in\mathbb{Z}}$ we define a double complex $A'_{\bullet,\bullet}$ with maps

$$d'^v_{p,q} = (-1)^p \varphi_{p,q} \qquad d'^h_{p,q} = d^h_{p,q}$$

But

$$d'^v_{p,q} = (-1)^p \varphi_{p,q} = (-1)^p(-1)^p d^v_{p,q} = d^v_{p,q}$$

i.e. $A'_{\bullet,\bullet} = A_{\bullet,\bullet}$ and thus the sign trick is a one to one correspondence between objects of $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ and double complexes.

(c) We can consider $C_\bullet$ (resp. $B[-1]_\bullet$) as a double complex with only one non trivial row in the following way

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & C_n \arrow[r, "d"] \arrow[u] & C_{n-1} \arrow[r, "d"] \arrow[u] & C_{n-2} \arrow[r] \arrow[u] & \dots \\
\dots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & B[-1]_n \arrow[r, "-d"] \arrow[u] & B[-1]_{n-1} \arrow[r, "-d"] \arrow[u] & B[-1]_{n-2} \arrow[r] \arrow[u] & \dots \\
\dots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

Now for the exactness of the short sequence, we denote

$$0 \longrightarrow C_\bullet \xrightarrow{\iota_{\bullet,\bullet}} D_{\bullet,\bullet} \xrightarrow{\pi_{\bullet,\bullet}} B[-1]_\bullet \longrightarrow 0$$

Setting the indices such that

$$C_{1,n} = C_n \quad D_{1,n} = C_n \quad D_{0,n} = B[-1]_{n+1} \quad B[-1]_{0,n} = B[-1]_n$$

And $\iota_{\bullet,\bullet}, \pi_{\bullet,\bullet}$ are defined as

$$\iota_{p,q} = \begin{cases} \mathrm{Id}_{C_q} & \text{if } p = 1 \\ \mathrm{Id}_0 & \text{if } p \neq 1 \end{cases}$$

$$\pi_{p,q} = \begin{cases} \mathrm{Id}_{B_q} & \text{if } p = 0 \\ \mathrm{Id}_0 & \text{if } p \neq 0 \end{cases}$$

From point b) this corresponds to a short sequence in $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ using the sign trick. From exercise 1, we only to check exactness at each non trivial rows i.e.

i.
$$\iota_{1,n} = \mathrm{Id}_{C_n} : C_{1,n} = C_n \to D_{1,n} = C_n$$
is an injective morphism of chain complexes.

ii.
$$\pi_{0,n} = \mathrm{Id}_{B_n} : D_{0,n} = B_n \to B[-1]_{n+1} = B_n$$
is a surjective morphism of chain complexes.

iii.
$$\iota_{1,n} \circ \pi_{1,n} = 0 \qquad \iota_{0,n} \circ \pi_{0,n} = 0$$

i. and ii. are clear from the definitions of $\iota_{1,n}$ and $\pi_{0,n}$.
Finally iii. holds since

$$\iota_{0,n} = 0 \ \forall n \in \mathbb{Z} \text{ and } \pi_{1,n} = 0 \ \forall n \in \mathbb{Z}$$

Note that the mapping cone of f : $D_\bullet := \mathrm{Tot}(D_{\bullet,\bullet})$ is given by :

$$D_\bullet = B[-1]_\bullet \oplus C_\bullet$$

with chain map

$$d_{D,\bullet} = \begin{pmatrix} -d_{B[-1],\bullet} & 0 \\ f[-1]_\bullet & d_{C,\bullet} \end{pmatrix}$$

We check that it is indeed a chain complex :

$$d_{D,n} \circ d_{D,n-1} = \begin{pmatrix} -d_{B[-1],n} & 0 \\ f[-1]_n & d_{C,n} \end{pmatrix} \begin{pmatrix} -d_{B[-1],n-1} & 0 \\ f[-1]_{n-1} & d_{C,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} d_{B[-1],n} \circ d_{B[-1],n-1} & 0 \\ -f[-1]_n \circ d_{B[-1],n-1} + d_{C,n} \circ f[-1]_{n-1} & d_{C,n} \circ d_{C,n-1} \end{pmatrix} = 0$$

since $f_\bullet : B_\bullet \to C_\bullet$ is a chain map.